Lecture 15

7.8 - Improper Integrals

In this section, we extend the concept of an integral to infinite lengths and to functions with an infinite discontinuity in the interval of integration.

Examples of these:

$$\int_1^\infty \frac{1}{x^3}\,dx, \quad \int_0^1 \frac{1}{x^3}\,dx, \quad \int_{-\infty}^\infty \frac{1}{4+x^2}\,dx$$

Let's deal first with infinite intervals:

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the integral exists and is finite.

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limit exists and is finite.

Terminology: The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called convergent if the corresponding limits exist and are finite, and are called divergent otherwise.

(c) If (for any value of c) both $\int_{-\infty}^{c} f(x)\,dx$ & $\int_{c}^{\infty} f(x)\,dx$ are convergent, then we define:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

Ex: Determine whether the following integrals converge or diverge:

ⓐ $\int_{1}^{\infty} \frac{1}{x}\,dx$ ⓑ $\int_{-\infty}^{5} e^{2x}\,dx$

ⓐ $\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \left(\ln|t| - \ln|1|\right)$

$= \lim_{t\to\infty} \ln|t| = \infty$ divergent

ⓑ $\int_{-\infty}^{5} e^{2x}\,dx = \lim_{t\to-\infty} \int_{t}^{5} e^{2x}\,dx = \lim_{t\to-\infty} \frac{1}{2}\left(e^{10} - e^{2t}\right)$

$= \frac{1}{2}\left(e^{10} - 0\right) = \frac{1}{2}e^{10}$ convergent

Ex: Determine whether $\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx$ is convergent or divergent. If it is convergent, find its value.

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{4+x^2}\,dx + \int_{0}^{\infty} \frac{1}{4+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{4+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{4+x^2}\,dx$$

$$= \lim_{t\to-\infty} \left(\tfrac{1}{2}\arctan(0) - \tfrac{1}{2}\arctan\left(\tfrac{t}{2}\right)\right) + \lim_{t\to\infty} \left(\tfrac{1}{2}\arctan\left(\tfrac{t}{2}\right) - \tfrac{1}{2}\arctan(0)\right)$$

$$= \left[0 - \left(\frac{-\pi}{4}\right)\right] + \left[\frac{\pi}{4} - 0\right] = \boxed{\frac{\pi}{2}}$$ convergent.

Theorem: $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ is convergent if $p>1$, divergent if $p \le 1$.

proof

$$\int_{1}^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x^p}\,dx = \begin{cases} \lim_{t\to\infty} \left(\ln|t| - \ln|1|\right), & p=1 \\ \lim_{t\to\infty} \left(\frac{1}{(1-p)t^{p-1}} - \frac{1}{1-p}\right), & p\neq 1 \end{cases}$$

$\underline{p=1}$: $\lim_{t\to\infty} \ln|t| = \infty$ divergent.

$\underline{p<1}$: then $p-1<0$ so $\lim_{t\to\infty} \frac{1}{(1-p)t^{p-1}} = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} = \infty$ divergent
$(1-p>0)$

$\underline{p>1}$: $p-1>0$ so $\lim_{t\to\infty} \frac{1}{(1-p)t^{p-1}} = 0$ convergent.

The next kind of improper integral is when the integrand has an infinite discontinuity in the interval of integration. This again has 3 cases, all of which are handled in the same way as before:

ⓐ If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if the limit exists and is finite.

ⓑ If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

if the limit exists and is finite.

Once again, we call the improper integral $\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exists and is finite, and is called divergent otherwise.

ⓒ If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)\,dx$ & $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example Determine whether $\int_0^2 \frac{1}{x-2}\,dx$ is convergent or divergent.

$\frac{1}{x-2}$ is discontinuous at $x=2$.

$$\int_0^2 \frac{1}{x-2}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-}\left(\ln|t-2| - \ln|0-2|\right)$$

$$= \underbrace{\lim_{t\to 2^-} \ln|t-2|}_{\to -\infty} - \ln 2$$

divergent

Theorem $\int_0^1 \frac{1}{x^p}\,dx$ is divergent if $p \geq 1$ & convergent if $p<1$. (The 1 is sort of irrelevant, as long as it is a # > 0)

(Notice this is backwards to before!)

proof: $$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^p}\,dx = \begin{cases} \ln|1| - \lim_{t\to 0^+}\ln|t|, & p=1 \\ \frac{1}{p+1} - \lim_{t\to 0^+}\frac{1}{(1-p)t^{p-1}}, & p\neq 1 \end{cases}$$

$p=1$: $\lim_{t\to 0^+} \ln|t| = -\infty$, divergent.

$p>1$: $p-1>0 \Rightarrow \lim_{t\to 0^+}\frac{1}{(1-p)t^{p-1}} = -\infty$, divergent.

$p<1$: $p-1<0 \Rightarrow \lim_{t\to 0^+}\frac{1}{(1-p)t^{p-1}} = \lim_{t\to 0^+}\frac{t^{1-p}}{1-p} = 0$, convergent.

Ex: Determine whether $\int_{-2}^{3} \frac{1}{(x-1)^4}\,dx$ is convergent or divergent. If it is convergent, give its value.

$$\int_{-2}^{3} \frac{1}{(x-1)^4}\,dx = \int_{-2}^{1} \frac{1}{(x-1)^4}\,dx + \int_{1}^{3} \frac{1}{(x-1)^4}\,dx$$

$$\int_{-2}^{1} \frac{1}{(x-1)^4}\,dx = \lim_{t\to 1^-} \int_{-2}^{t} \frac{1}{(x-1)^4}\,dx = \lim_{t\to 1^-} \left(\frac{-1}{3}\left(\frac{1}{(t-1)^3} + \frac{1}{27}\right)\right) = -\infty$$

divergent

Since this piece is divergent, the whole integral is.

($\int_{1}^{3} \frac{1}{(x-1)^4}\,dx$ is also divergent.)

Comparison Test for Integrals

If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

ⓐ If $\int_{a}^{\infty} f(x)\,dx$ is convergent, then $\int_{a}^{\infty} g(x)\,dx$ is convergent

ⓑ If $\int_{a}^{\infty} g(x)\,dx$ is divergent, then $\int_{a}^{\infty} f(x)\,dx$ is divergent.

Ex: Use the comparison test to determine whether the following integrals are convergent or divergent.

ⓐ $\int_1^\infty \frac{1}{x^2+x+1}\,dx$ ⓑ $\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx$ ⓒ $\int_0^\pi \frac{\cos^2 x}{\sqrt{x}}\,dx$

ⓓ $\int_0^\infty \frac{e^{-x}}{1+\sin^2 x}\,dx$

ⓐ For $x \geq 1$, $\frac{1}{x^2+x+1} \leq \frac{1}{x^2}$, so by our first theorem: $\int_1^\infty \frac{1}{x^2}\,dx$ converges, thus $\int_1^\infty \frac{1}{x^2+x+1}\,dx$ also converges by the comparison test.

ⓑ For $x \geq 1$, $\frac{1}{x-\frac{1}{2}} \geq \frac{1}{x}$

Since $\int_1^\infty \frac{dx}{x}$ diverges, so does $\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx$.

ⓒ $\cos^2 x \leq 1$ always, so $\frac{\cos^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$. Our 2^{nd} theorem says $\int_0^\pi \frac{1}{\sqrt{x}}\,dx$ converges, so the comparison test says $\int_0^\pi \frac{\cos^2 x}{\sqrt{x}}\,dx$ converges.

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ⓓ Since $1+\sin^2 x \geq 1$, $\frac{e^{-x}}{1+\sin^2 x} \leq e^{-x}$

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left(-e^{-t}+e^{0}\right)$$

$$= 0+1 = 1$$ convergent

So, by the comparison test, $\int_0^\infty \frac{e^{-x}}{1+\sin^2 x}\,dx$

converges.